

# A NECESSARY FLEXIBILITY CONDITION OF A NONDEGENERATE SUSPENSION IN LOBACHEVSKY 3-SPACE

Dmitriy Slutskiy \*

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## Abstract

We show that some combination of the lengths of all edges of the equator of a flexible suspension in Lobachevsky 3-space is equal to zero (each length is taken either positive or negative in this combination).

## Keywords

flexible polyhedron, Lobachevsky space, hyperbolic space, flexible suspension, Connelly method, equator of suspension, length of edge.

## 1 Introduction

A polyhedron (more precisely, a polyhedral surface) is said to be flexible if its spatial shape can be changed continuously due to changes of its dihedral angles only, i.e., if every face remains congruent to itself during the flex.

In 1897 R. Bricard [1] described all flexible octahedra in Euclidean 3-space. The Bricard's octahedra were the first examples of flexible polyhedra (with self-intersections). Bricard's octahedra are special cases of Euclidean flexible suspensions. In 1974 R. Connelly [2] proved that some combination of the lengths of all edges of the equator of a flexible suspension in Euclidean 3-space is equal to zero (each length is taken either positive or negative in this combination). The method applied by R. Connelly, is to reduce the problem to the study of an analytic function of complex variable in neighborhoods of its singular points.

In 2001 S. N. Mikhalev [3] reproved the above-mentioned result of R. Connelly by algebraic methods. Moreover, S. N. Mikhalev proved that for every spatial quadrilateral formed by edges of a flexible suspension and containing its both poles there is a combination of the lengths (taken either positive or negative) of the edges of the quadrilateral, which is equal to zero.

The aim of this work is to prove a similar result for the equator of a flexible suspension in Lobachevsky 3-space, applying the method of Connelly [2].

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## 2 Formulating the main result

Let  $\mathcal{K}$  be a simplicial complex. A *polyhedron* (a *polyhedral surface*) in Lobachevsky 3-space is a continuous map from  $\mathcal{K}$  to  $\mathbb{H}^3$ , which sends every  $k$ -dimensional simplex of  $\mathcal{K}$  into a subset of a  $k$ -dimensional plane of Lobachevsky space ( $k \leq 2$ ). Images of topological 2-simplices are called faces, images of topological 1-simplices are called edges and images of topological 0-simplices are called vertices of the polyhedron. Note that in our definition an image of a simplex can be degenerate (for instance, a face can lie on a straight hyperbolic line, and an edge can be reduced to one point), and faces can intersect in their interior points. If  $v_1, \dots, v_W$  are the vertices of  $\mathcal{K}$ , and if  $\mathcal{P} : \mathcal{K} \rightarrow \mathbb{H}^3$  is a polyhedron, then  $\mathcal{P}$  is determined by  $W$  points  $P_1, \dots, P_W \in \mathbb{H}^3$ , where  $P_j \stackrel{\text{def}}{=} \mathcal{P}(v_j)$ ,  $j = 1, \dots, W$ .

If  $\mathcal{P} : \mathcal{K} \rightarrow \mathbb{H}^3$  and  $\mathcal{Q} : \mathcal{K} \rightarrow \mathbb{H}^3$  are two polyhedra, then we say  $\mathcal{P}$  and  $\mathcal{Q}$  are *congruent* if there exists a motion  $\mathcal{A} : \mathbb{H}^3 \rightarrow \mathbb{H}^3$  such that  $\mathcal{Q} = \mathcal{A} \circ \mathcal{P}$  (i.e. the isometric mapping  $\mathcal{A}$  sends every vertex of  $\mathcal{P}$  into a corresponding vertex of  $\mathcal{Q}$ :  $Q_j = \mathcal{A}(P_j)$ , or in other words  $\mathcal{Q}(v_j) = \mathcal{A}(\mathcal{P}(v_j))$ ,  $j = 1, \dots, W$ ). We say  $\mathcal{P}$  and  $\mathcal{Q}$  are *isometric (in the intrinsic metric)* if each edge of  $\mathcal{P}$  has the same length as the corresponding edge of  $\mathcal{Q}$ , i.e. if  $\langle v_j, v_k \rangle$  is a 1-simplex of  $\mathcal{K}$  then  $d_{\mathbb{H}^3}(Q_j, Q_k) = d_{\mathbb{H}^3}(P_j, P_k)$ , where  $d_{\mathbb{H}^3}(\cdot, \cdot)$  stands for the distance in Lobachevsky space  $\mathbb{H}^3$ .

A polyhedron  $\mathcal{P}$  is *flexible* if, for some continuous one parameter family of polyhedra  $\mathcal{P}_t : \mathcal{K} \rightarrow \mathbb{H}^3$ ,  $0 \leq t \leq 1$ , the following three conditions hold true: (1)  $\mathcal{P}_0 = \mathcal{P}$ ; (2) each  $\mathcal{P}_t$  is isometric to  $\mathcal{P}_0$ ; (3) some  $\mathcal{P}_t$  is not congruent to  $\mathcal{P}_0$ .

Let  $\mathcal{K}$  be defined as follows:  $\mathcal{K}$  has vertices  $v_0, v_1, \dots, v_V, v_{V+1}$ , where  $v_1, \dots, v_V$  form a cycle ( $v_j$  adjacent to  $v_{j+1}$ ,  $j = 1, \dots, V-1$ , and  $v_V$  adjacent to  $v_1$ ), and  $v_0$  and  $v_{V+1}$  are each adjacent to all of  $v_1, \dots, v_V$ . Each polyhedron  $\mathcal{P}$  based on  $\mathcal{K}$  is called a *suspension*. Call  $N \stackrel{\text{def}}{=} \mathcal{P}(v_0)$  the north pole, and  $S \stackrel{\text{def}}{=} \mathcal{P}(v_{V+1})$  the south pole, and  $P_j \stackrel{\text{def}}{=} \mathcal{P}(v_j)$ ,  $j = 1, \dots, V$  vertices of the *equator*  $\mathcal{P}$ .

Assume that a suspension  $\mathcal{P}$  is flexible. If we suppose the segment  $NS$  to be an extra edge, then  $\mathcal{P}$  becomes a set of  $V$  tetrahedra glued cyclically along their common edge  $NS$ . We call a suspension *nondegenerate* if none of these tetrahedra lies on a hyperbolic 2-plane. Note that a nondegenerate suspension  $\mathcal{P}$  does not flex if the distance between  $N$  and  $S$  remains constant. Therefore, as in the Euclidean case [2] we assume that the length of  $NS$  is variable during the flex of  $\mathcal{P}$ . Examples of degenerate suspensions are a double covered cap — a suspension with coinciding poles (see Fig. 1), and a suspension with a wing — a suspension whose vertices  $N$ ,  $S$ ,  $P_{i-1}$ , and  $P_{i+1}$  lie on a straight line for some  $i$  (see Fig. 2). In this paper we will not study the degenerate flexible suspensions.

The main result of the paper is

**Theorem 1** *Let  $\mathcal{P}$  be a nondegenerate flexible suspension in Lobachevsky 3-space with the poles  $S$  and  $N$ , and with the vertices of the equator  $P_j$ ,  $j = 1, \dots, V$ . Then for some set of signs  $\sigma_{j,j+1} \in \{+1, -1\}$ ,  $j = 1, \dots, V$ , the combination of the lengths  $e_{j,j+1}$  of all edges  $P_j P_{j+1}$  of the equator of  $\mathcal{P}$  taken with the corresponding signs  $\sigma_{j,j+1}$  is equal to*

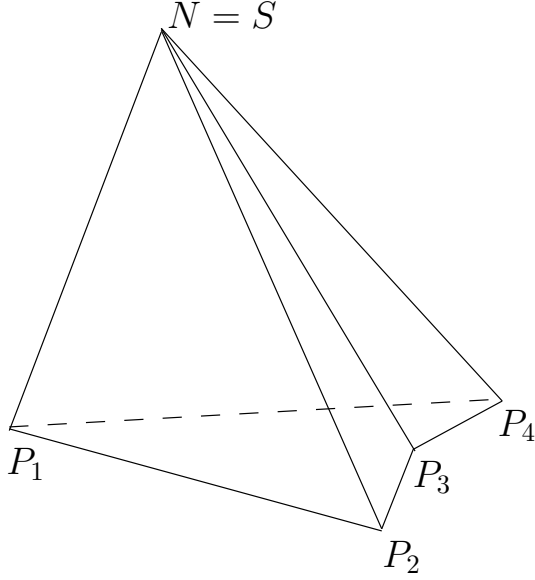


Figure 1: A double covered cap.

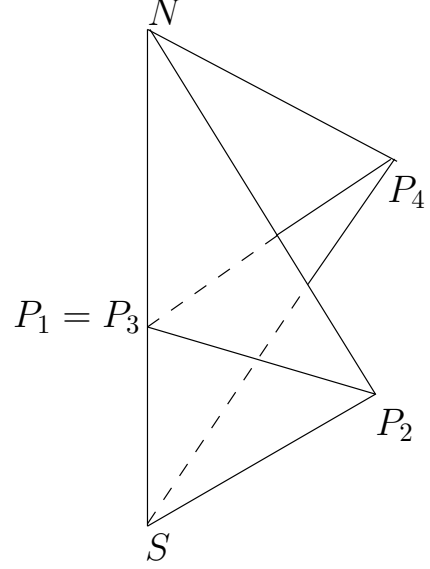


Figure 2: A suspension with a wing.

zero, i.e.

$$\sum_{j=1}^V \sigma_{j,j+1} e_{j,j+1} = 0. \quad (1)$$

(Here and below, by definition, it is considered that  $P_{V+1} \stackrel{\text{def}}{=} P_1$ ,  $P_V P_{V+1} \stackrel{\text{def}}{=} P_V P_1$ ,  $\sigma_{V,V+1} \stackrel{\text{def}}{=} \sigma_{V,1}$ , and  $e_{V,V+1} \stackrel{\text{def}}{=} e_{V,1}$ .)

### 3 Connolly's equation of flexibility of a suspension

R. Connolly in [2] obtained an equation of flexibility of a nondegenerate suspension in Euclidean 3-space. Following him, in this section we will obtain an equation of flexibility of a nondegenerate suspension in Lobachevsky 3-space.

Let us place a nondegenerate suspension  $\mathcal{P}$  into the Poincaré upper half-space model [4] of Lobachevsky 3-space  $\mathbb{H}^3$  in such a way that the poles  $N$  and  $S$  of  $\mathcal{P}$  lie on the axis  $Oz$  of the Cartesian coordinate system of the Poincaré model (see Fig. 3). Let  $S$  has the coordinates  $(0, 0, z_S)$ ,  $N$  has the coordinates  $(0, 0, z_N)$ , and  $P_j$  has the coordinates  $(x_j, y_j, z_j)$ ,  $j = 1, \dots, V$ . Also we denote the length of the edge  $NP_j$  by  $e_j$ , and the length of  $SP_j$  by  $e'_j$ ,  $j = 1, \dots, V$ .

Consider a Euclidean orthogonal projection  $\tilde{\mathcal{P}}$  of  $\mathcal{P}$  on the plane  $Oxy$  (see Fig. 4). Also  $\tilde{\mathcal{P}}$  is a hyperbolic projection of  $\mathcal{P}$  on  $Oxy$  from the only point at infinity of  $\mathbb{H}^3$  which does not lie on  $Oxy$ . This projection sends poles  $N$  and  $S$  of  $\mathcal{P}$  to the origin  $O$   $(0, 0)$  on the plane  $Oxy$ ,  $P_j$  to the point  $\tilde{P}_j$   $(x_j, y_j)$ , edges  $NP_j$  and  $SP_j$  to the Euclidean segment  $O\tilde{P}_j$ , and the edge  $P_j P_{j+1}$  of the equator of  $\mathcal{P}$  to the Euclidean segment  $\tilde{P}_j \tilde{P}_{j+1}$ ,  $j = 1, \dots, V$  (here and below  $\tilde{P}_{V+1} \stackrel{\text{def}}{=} \tilde{P}_1$ ,  $x_{V+1} \stackrel{\text{def}}{=} x_1$ ,  $y_{V+1} \stackrel{\text{def}}{=} y_1$ ,  $z_{V+1} \stackrel{\text{def}}{=} z_1$ ).

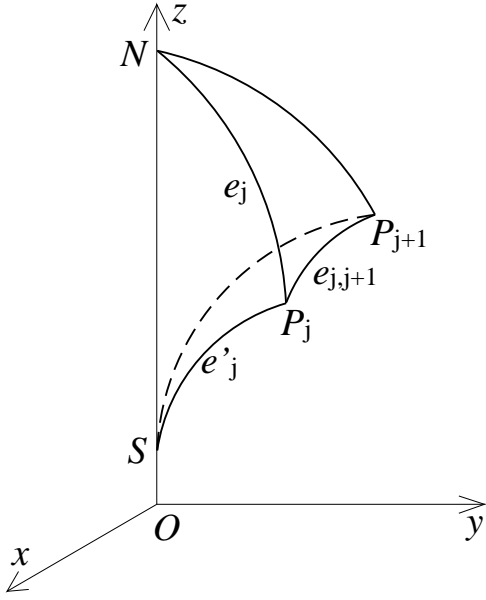


Figure 3: A fragment of the lateral surface of  $\mathcal{P}$ .

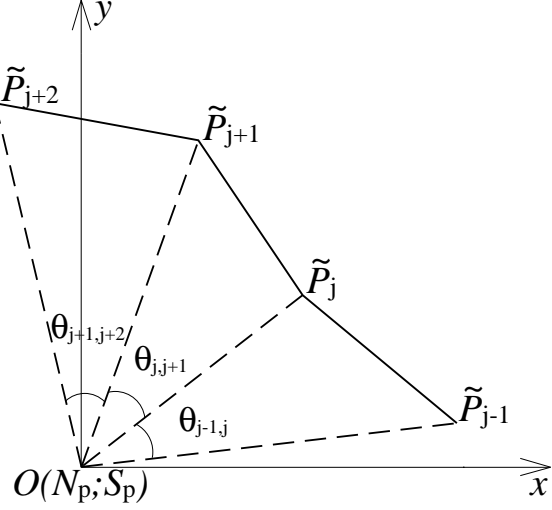


Figure 4: A projection of  $\mathcal{P}$  on  $Oxy$ .

Polar coordinates  $(\rho_j, \theta_j)$  of  $\tilde{P}_j$ ,  $j = 1, \dots, V$ , are related to its Cartesian coordinates by the formulas (see Fig. 5):

$$\rho_j = \sqrt{x_j^2 + y_j^2}, \quad \sin \theta_j = \frac{y_j}{\rho_j} = \frac{y_j}{\sqrt{x_j^2 + y_j^2}}, \quad \cos \theta_j = \frac{x_j}{\rho_j} = \frac{x_j}{\sqrt{x_j^2 + y_j^2}}. \quad (2)$$

Note that by construction, the dihedral angle  $\theta_{j,j+1}$  of the tetrahedron  $NSP_jP_{j+1}$  at the edge  $NS$  is equal to the flat angle  $\angle \tilde{P}_j O \tilde{P}_{j+1}$ ,  $j = 1, \dots, V$ , and

$$\theta_{j,j+1} = \theta_{j+1} - \theta_j. \quad (3)$$

Note as well that the value of  $\theta_{j,j+1}$  can be negative. Applying the trigonometric ratio of the difference of two angles and (3), we get:

$$\cos \theta_{j,j+1} = \cos \theta_{j+1} \cos \theta_j + \sin \theta_{j+1} \sin \theta_j, \quad \sin \theta_{j,j+1} = \sin \theta_{j+1} \cos \theta_j - \cos \theta_{j+1} \sin \theta_j. \quad (4)$$

Taking into account (2) we reduce (4) to

$$\cos \theta_{j,j+1} = \frac{x_j x_{j+1} + y_j y_{j+1}}{\sqrt{x_{j+1}^2 + y_{j+1}^2} \sqrt{x_j^2 + y_j^2}}, \quad \sin \theta_{j,j+1} = \frac{x_j y_{j+1} - y_j x_{j+1}}{\sqrt{x_{j+1}^2 + y_{j+1}^2} \sqrt{x_j^2 + y_j^2}}.$$

Then, according to Euler's formula,

$$e^{i\theta_{j,j+1}} = \cos \theta_{j,j+1} + i \sin \theta_{j,j+1} = \frac{(x_j x_{j+1} + y_j y_{j+1}) + i(x_j y_{j+1} - y_j x_{j+1})}{\sqrt{x_{j+1}^2 + y_{j+1}^2} \sqrt{x_j^2 + y_j^2}}. \quad (5)$$

Following R. Connelly [2], we remark that the sum of the dihedral angles  $\theta_{j,j+1}$  of all tetrahedra  $NSP_jP_{j+1}$ ,  $j = 1, \dots, V$ , at the edge  $NS$  is constant and a multiple of  $2\pi$  (here

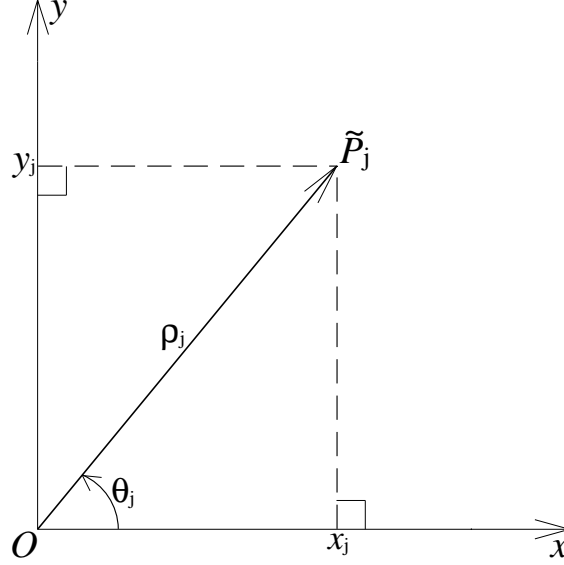


Figure 5: The coordinates of  $\tilde{P}_j$ .

and below  $\theta_{V,V+1} \stackrel{\text{def}}{=} \theta_{V,1}$ ,  $\theta_{V+1} \stackrel{\text{def}}{=} \theta_1$ ,  $\rho_{V+1} \stackrel{\text{def}}{=} \rho_1$ ), i.e.

$$\sum_{j=1}^V \theta_{j,j+1} = 2\pi m \quad \text{for some integer } m, \quad (6)$$

and remains so during the deformation of the suspension, when the values of the angles  $\theta_{j,j+1}$ ,  $j = 1, \dots, V$ , vary continuously.

We rewrite the equation of flexibility (6) in a convenient form:

$$\prod_{j=1}^V e^{i\theta_{j,j+1}} = 1. \quad (7)$$

Thus, taking into account (5), we see that coordinates of vertices of  $\mathcal{P}$  are related as follows:

$$\prod_{j=1}^V \frac{(x_j x_{j+1} + y_j y_{j+1}) + i(x_j y_{j+1} - y_j x_{j+1})}{x_j^2 + y_j^2} = 1, \quad (8)$$

or in other notations

$$\prod_{j=1}^V F_{j,j+1} = \prod_{j=1}^V \frac{G_{j,j+1}}{\rho_j \rho_{j+1}} = \prod_{j=1}^V \frac{G_{j,j+1}}{\rho_j^2} = 1, \quad (9)$$

where  $G_{j,m} = (x_j x_m + y_j y_m) + i(x_j y_m - y_j x_m)$ ,  $F_{j,m} = \frac{G_{j,m}}{\rho_j \rho_m}$ ,  $j, m = 1, \dots, V$ , and  $G_{V,V+1} \stackrel{\text{def}}{=} G_{V,1}$ ,  $F_{V,V+1} \stackrel{\text{def}}{=} F_{V,1}$ .

When studying the deformation  $\mathcal{P}_t$  of the suspension  $\mathcal{P}$ , all objects and values related to  $\mathcal{P}_t$  naturally succeed from the notations for the corresponding entities related to  $\mathcal{P}$ . For example, the coordinate  $x_j(t)$  of the point  $P_j(t)$  of the deformation  $\mathcal{P}_t$  corresponds to the coordinate  $x_j$  of the point  $P_j$  of the suspension  $\mathcal{P}$ , the dihedral angle  $\theta_{j,j+1}(t)$  of the tetrahedron  $N(t)S(t)P_j(t)P_{j+1}(t)$  at the edge  $N(t)S(t)$  corresponds to the dihedral angle  $\theta_{j,j+1}$  of the tetrahedron  $NSP_jP_{j+1}$  at the edge  $NS$ , etc.

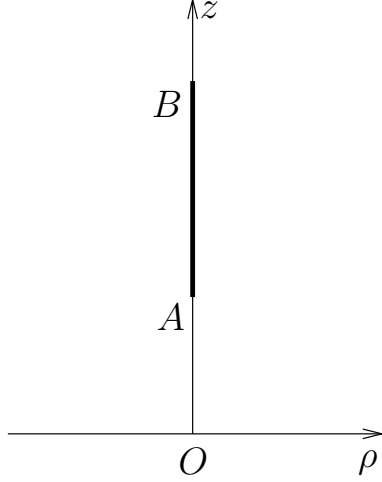


Figure 6: Points on a plane in the lemma 1.

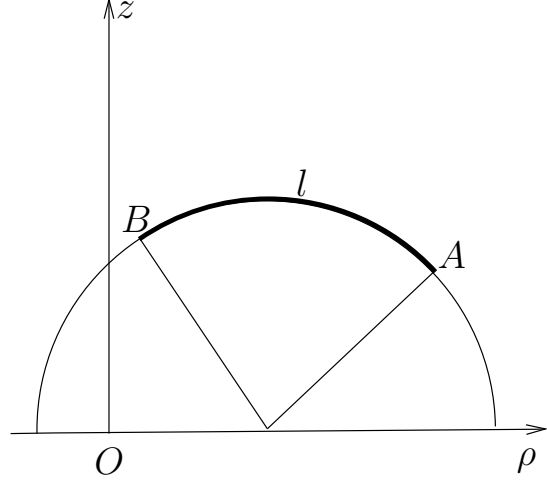


Figure 7: Points on a plane in the lemma 2.

## 4 The equation of flexibility of a suspension in terms of the lengths of its edges

In this section we are going to express the equation of flexibility of a suspension (8) in terms of the lengths of edges of  $\mathcal{P}$ . Recall that the lengths of the edges of  $\mathcal{P}$  remain constant during the flex. To this purpose we need to demonstrate the truth of two following statements. The first of them can be verified by direct calculation (see also Fig. 6).

**Lemma 1** *Given a Poincaré upper half-plane  $\mathbb{H}^2$  with the coordinates  $(\rho, z)$  (i.e., with the metric given by the formula  $ds^2 = \frac{d\rho^2 + dz^2}{z^2}$ ). Then the distance between the points  $A (\rho_0, z_A)$  and  $B (\rho_0, z_B)$ , having the same first coordinate  $\rho_0$ , is calculated by the formula*

$$d_{\mathbb{H}^2}(A, B) = \left| \ln \frac{z_B}{z_A} \right|. \quad (10)$$

**Lemma 2** *Given a Poincaré upper half-plane  $\mathbb{H}^2$  with the coordinates  $(\rho, z)$  (i.e., with the metric given by the formula  $ds^2 = \frac{d\rho^2 + dz^2}{z^2}$ ). Then the distance  $l \stackrel{\text{def}}{=} d_{\mathbb{H}^2}(A, B)$  between the points  $A (\rho_A, z_A)$  and  $B (\rho_B, z_B)$  is related to their coordinates by the formula*

$$(\rho_B - \rho_A)^2 + z_A^2 + z_B^2 = 2z_A z_B \cosh l. \quad (11)$$

**Proof.** According to the part (2) of the Corollary A.5.8 [5], the distance between the points with the coordinates  $(x, t)$  and  $(y, s)$  in the Poincaré upper half-space model  $\mathbb{R}^n \times \mathbb{R}^+$  of Lobachevsky  $(n + 1)$ -space  $\mathbb{H}^{n+1}$  is calculated by the formula

$$d_{\mathbb{H}^{n+1}}((x, t), (y, s)) = 2 \operatorname{artanh} \left( \frac{\|x - y\|^2 + (t - s)^2}{\|x - y\|^2 + (t + s)^2} \right)^{1/2}, \quad (12)$$

where the symbol  $\|\cdot\|$  stands for the standard Euclidean norm in  $\mathbb{R}^n$ .

By (12) the distance between the points  $A$  and  $B$  (see Fig. 7) is calculated by the formula

$$l = 2 \operatorname{artanh} \left( \frac{(\rho_A - \rho_B)^2 + (z_A - z_B)^2}{(\rho_A - \rho_B)^2 + (z_A + z_B)^2} \right)^{1/2}, \quad (13)$$

where  $n = 1$ ,  $(x, t) = (\rho_A, z_A)$  and  $(y, s) = (\rho_B, z_B)$ .

After a series of transformations of the formula (13) we get:

$$(\rho_A - \rho_B)^2 \left( \cosh^2 \frac{l}{2} - \sinh^2 \frac{l}{2} \right) + (z_A^2 + z_B^2) \left( \cosh^2 \frac{l}{2} - \sinh^2 \frac{l}{2} \right) = 2z_A z_B \left( \cosh^2 \frac{l}{2} + \sinh^2 \frac{l}{2} \right). \quad (14)$$

By two identities of hyperbolic geometry,  $\cosh^2 \frac{l}{2} - \sinh^2 \frac{l}{2} = 1$  and  $\cosh l = \cosh^2 \frac{l}{2} + \sinh^2 \frac{l}{2}$ , (14) reduces to (11).  $\square$

Let us express  $G_{j,j+1}$  and  $\rho_j^2$  in terms of the length of edges of  $\mathcal{P}$ .

We assume that the coordinates of the south pole  $S$  are  $(0, 0, 1)$ . Let  $t \stackrel{\text{def}}{=} e^{d_{\mathbb{H}^3}(N, S)}$ , where  $d_{\mathbb{H}^3}(N, S)$  is the distance between the poles  $N$  and  $S$  of  $\mathcal{P}$ . Without loss of generality, we assume that  $z_N \geq z_S$ . Then, by Lemma 1, the coordinates of  $N$  are  $(0, 0, t)$ .

Applying Lemma 2 to the points  $S$  and  $P_j$  lying on the hyperbolic plane  $SNP_j$ , by the formula (11) we get:

$$\rho_j^2 + z_j^2 + 1 = 2z_j \cosh e'_j. \quad (15)$$

Now we apply Lemma 2 to the vertices  $N$  and  $P_j$ :

$$\rho_j^2 + z_j^2 + t^2 = 2tz_j \cosh e_j. \quad (16)$$

Subtracting (15) from (16), under the assumption that  $t \cosh e_j \neq \cosh e'_j$ , we get:

$$z_j = \frac{t^2 - 1}{2(t \cosh e_j - \cosh e'_j)}. \quad (17)$$

Also, taking into account (15) and (17), we obtain:

$$\rho_j^2 = 2z_j \cosh e'_j - z_j^2 - 1 = \frac{(t^2 - 1) \cosh e'_j}{(t \cosh e_j - \cosh e'_j)} - \frac{(t^2 - 1)^2}{4(t \cosh e_j - \cosh e'_j)^2} - 1. \quad (18)$$

Let  $\rho_{j,j+1}$  denote the Euclidean distance between the points  $\tilde{P}_j$  and  $\tilde{P}_{j+1}$ ,  $j = 1, \dots, V$  (here and below  $\rho_{V,V+1} \stackrel{\text{def}}{=} \rho_{V,1}$ ). Applying Lemma 2 to the vertices  $P_j$  and  $P_{j+1}$ , we get:

$$\rho_{j,j+1}^2 = 2z_j z_{j+1} \cosh e_{j,j+1} - z_j^2 - z_{j+1}^2. \quad (19)$$

By the Pythagorean theorem  $\rho_{j,j+1}$  is related to the Cartesian coordinates of  $\tilde{P}_j$  and  $\tilde{P}_{j+1}$  by the formula

$$\rho_{j,j+1} = \sqrt{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2}. \quad (20)$$

By (2) the equation (20) reduces to:

$$\rho_{j,j+1}^2 = (x_j^2 + y_j^2) + (x_{j+1}^2 + y_{j+1}^2) - 2(x_j x_{j+1} + y_j y_{j+1}) = \rho_j^2 + \rho_{j+1}^2 - 2(x_j x_{j+1} + y_j y_{j+1}).$$

Thus, taking into account (18) and (19), the expression  $x_j x_{j+1} + y_j y_{j+1}$ , which is a part of  $G_{j,j+1}$  from (9), is related to the lengths of edges of  $\mathcal{P}$  by the formula

$$x_j x_{j+1} + y_j y_{j+1} = \frac{\rho_j^2 + \rho_{j+1}^2 - \rho_{j,j+1}^2}{2} = z_j \cosh e'_j + z_{j+1} \cosh e'_{j+1} - z_j z_{j+1} \cosh e_{j,j+1} - 1. \quad (21)$$

Substituting (17) in (21) we get:

$$\begin{aligned} x_j x_{j+1} + y_j y_{j+1} = & \frac{1}{2} \left( \frac{(t^2 - 1) \cosh e'_j}{(t \cosh e_j - \cosh e'_j)} + \frac{(t^2 - 1) \cosh e'_{j+1}}{(t \cosh e_{j+1} - \cosh e'_{j+1})} - \right. \\ & \left. - \frac{(t^2 - 1)^2 \cosh e_{j,j+1}}{2(t \cosh e_j - \cosh e'_j)(t \cosh e_{j+1} - \cosh e'_{j+1})} - 2 \right). \end{aligned} \quad (22)$$

Let us now express  $x_j y_{j+1} - y_j x_{j+1}$ , which is also a part of  $G_{j,j+1}$ , in terms of the length of edges of  $\mathcal{P}$ .

According to (5) we know that

$$\cos \theta_{j,j+1} = \frac{x_j x_{j+1} + y_j y_{j+1}}{\rho_j \rho_{j+1}} \quad \text{and} \quad \sin \theta_{j,j+1} = \frac{x_j y_{j+1} - y_j x_{j+1}}{\rho_j \rho_{j+1}}. \quad (23)$$

Note that by definition (2),  $\rho_j > 0$ ,  $j = 1, \dots, V$ .

By the Pythagorean trigonometric identity, the formula

$$\sin \theta_{j,j+1} = \sigma_{j,j+1} \sqrt{1 - \cos^2 \theta_{j,j+1}} \quad (24)$$

holds true, where  $\sigma_{j,j+1} = 1$  if  $\sin \theta_{j,j+1} \geq 0$ , and  $\sigma_{j,j+1} = -1$  if  $\sin \theta_{j,j+1} < 0$  (remind that  $\theta_{j,j+1}$  is determined in (3)). Then (23) and (24) imply

$$\begin{aligned} x_j y_{j+1} - y_j x_{j+1} &= \rho_j \rho_{j+1} \sin \theta_{j,j+1} = \sigma_{j,j+1} \rho_j \rho_{j+1} \sqrt{1 - \cos^2 \theta_{j,j+1}} = \\ &= \sigma_{j,j+1} \rho_j \rho_{j+1} \sqrt{1 - \frac{(x_j x_{j+1} + y_j y_{j+1})^2}{\rho_j^2 \rho_{j+1}^2}} = \sigma_{j,j+1} \sqrt{\rho_j^2 \rho_{j+1}^2 - (x_j x_{j+1} + y_j y_{j+1})^2}. \end{aligned} \quad (25)$$

Substituting (18) and (22) in (25) we get

$$\begin{aligned} x_j y_{j+1} - y_j x_{j+1} &= \sigma_{j,j+1} \left[ \left( \frac{(t^2 - 1) \cosh e'_j}{(t \cosh e_j - \cosh e'_j)} - \frac{(t^2 - 1)^2}{4(t \cosh e_j - \cosh e'_j)^2} - 1 \right) \times \right. \\ &\times \left( \frac{(t^2 - 1) \cosh e'_{j+1}}{(t \cosh e_{j+1} - \cosh e'_{j+1})} - \frac{(t^2 - 1)^2}{4(t \cosh e_{j+1} - \cosh e'_{j+1})^2} - 1 \right) - \frac{1}{4} \left( \frac{(t^2 - 1) \cosh e'_j}{(t \cosh e_j - \cosh e'_j)} + \right. \\ &\left. \left. + \frac{(t^2 - 1) \cosh e'_{j+1}}{(t \cosh e_{j+1} - \cosh e'_{j+1})} - \frac{(t^2 - 1)^2 \cosh e_{j,j+1}}{2(t \cosh e_j - \cosh e'_j)(t \cosh e_{j+1} - \cosh e'_{j+1})} - 2 \right)^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (26)$$

Substituting (18), (22), and (26) in (8) we obtain the equation of flexibility of a suspension in terms of the lengths of edges of  $\mathcal{P}$ .



## 5 Proof of the theorem

In order to prove the theorem 1 we shall study singular points of the equation of flexibility of a suspension.

Assume that a nondegenerate suspension  $\mathcal{P}$  flexes. Then, as we have already mentioned in the section 2, the distance  $l_{NS}$  between the poles of  $\mathcal{P}$  changes during the flex. Let  $t \stackrel{\text{def}}{=} e^{l_{NS}}$  be the parameter of the flex of  $\mathcal{P}$ . The identity (9) holds true at every moment  $t$  of the flex, as the values of the expressions  $F_{j,j+1}$ ,  $G_{j,j+1}$ ,  $\rho_j^2$ ,  $j = 1, \dots, V$ , which make part (9), vary as  $t$  changes. Here the functions  $G_{j,j+1}(t) = [x_j x_{j+1} + y_j y_{j+1}](t) + i[x_j y_{j+1} - y_j x_{j+1}](t)$  and  $\rho_j^2(t)$ ,  $j = 1, \dots, V$ , are determined in (18), (22) and (26).

Assume now that for some  $j \in \{1, \dots, V\}$  the dihedral angle  $\theta_{j,j+1}(t)$  remains constant (the value of  $\theta_{j,j+1}(t)$  can also be equal to zero) as  $t$  changes. In this case the length of the edge  $N(t)S(t)$  of the tetrahedron  $N(t)S(t)P_j(t)P_{j+1}(t)$  must be constant as well (all other edges of the tetrahedron are also the edges of  $\mathcal{P}_t$ , therefore their lengths are fixed), i.e. the value of  $t$  does not change. As we mentioned in the section 2, in this case  $\mathcal{P}$  can not be flexible. Thus we have the contradiction. Therefore, the values of the angles  $\theta_{j,j+1}(t)$ ,  $j = 1, \dots, V$ , change continuously during the flex. Hence, there exists such an interval  $(t_1, t_2)$  that for all  $t \in (t_1, t_2)$  it is true that  $\theta_{j,j+1}(t) \neq 0$  for every  $j \in \{1, \dots, V\}$ .

We extend both sides of the equation of flexibility (9) as functions in  $t$  on the whole complex plane  $\mathbb{C}$ . By the theorem on the uniqueness of the analytic function [6], the expression (9) remains valid.

Analytic functions  $F_{j,j+1}(t)$ ,  $j = 1, \dots, V$ , have a finite number of algebraic singular points. Without loss of generality we can assume that none of these points lies in the interval  $(t_1, t_2)$ . For every  $F_{j,j+1}(t)$ ,  $j = 1, \dots, V$ , we choose a single-valued branch  $(F_{j,j+1}(t), D)$ , where  $D \subset \mathbb{C}$  is an unbounded domain containing  $(t_1, t_2)$ . Let  $\mathcal{W} \subset D$  be a path connecting  $t_0 \in (t_1, t_2)$  and  $\infty$ , such that  $t_0$  is a unique real point of  $\mathcal{W}$ . Let us calculate the limit of  $F_{j,j+1}(t)$  as  $t \rightarrow \infty$  along  $\mathcal{W}$ .

Taking into account (18) we get

$$\lim_{t \rightarrow \infty} \frac{\rho_j^2(t)}{t^2} = \lim_{t \rightarrow \infty} \left[ \frac{1}{t^2} \left( \frac{(t^2 - 1) \cosh e'_j}{(t \cosh e_j - \cosh e'_j)} - \frac{(t^2 - 1)^2}{4(t \cosh e_j - \cosh e'_j)^2} - 1 \right) \right] = -\frac{1}{4 \cosh^2 e_j}. \quad (27)$$

Similarly, from (22) we derive that

$$\lim_{t \rightarrow \infty} \frac{(x_j x_{j+1} + y_j y_{j+1})(t)}{t^2} = -\frac{\cosh e_{j,j+1}}{4 \cosh e_j \cosh e_{j+1}}. \quad (28)$$

Also from (25) and taking into account (27) and (28) we have:

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{(x_j y_{j+1} - y_j x_{j+1})^2(t)}{t^4} &= \lim_{t \rightarrow \infty} \left[ \frac{\rho_j^2(t) \rho_{j+1}^2(t) - (x_j x_{j+1} + y_j y_{j+1})^2(t)}{t^4} \right] = \\ &= \frac{1}{16 \cosh^2 e_j \cosh^2 e_{j+1}} - \frac{\cosh^2 e_{j,j+1}}{16 \cosh^2 e_j \cosh^2 e_{j+1}} = \frac{1 - \cosh^2 e_{j,j+1}}{16 \cosh^2 e_j \cosh^2 e_{j+1}}. \end{aligned}$$

Hence,

$$\lim_{t \rightarrow \infty} \frac{(x_j y_{j+1} - y_j x_{j+1})(t)}{t^2} = i \sigma_{j,j+1} \frac{\sqrt{\cosh^2 e_{j,j+1} - 1}}{4 \cosh e_j \cosh e_{j+1}}, \quad (29)$$

where  $\sigma_{j,j+1} \in \{+1, -1\}$  is determined by the single-valued branch  $(F_{j,j+1}(t), D)$  and by the path  $\mathcal{W}$ .

By definition of  $G_{j,j+1}(t)$  and according to (28) and (29), we get:

$$\lim_{t \rightarrow \infty} \frac{G_{j,j+1}(t)}{t^2} = - \frac{\cosh e_{j,j+1} + \sigma_{j,j+1} \sqrt{\cosh^2 e_{j,j+1} - 1}}{4 \cosh e_j \cosh e_{j+1}}. \quad (30)$$

By (30) and (27), the limit of the left-hand side of (9) at  $t \rightarrow \infty$

$$\lim_{t \rightarrow \infty} \prod_{j=1}^V F_{j,j+1}(t) = \lim_{t \rightarrow \infty} \prod_{j=1}^V \frac{F_{j,j+1}(t)/t^2}{\rho_j^2(t)/t^2} = \prod_{j=1}^V \left( \cosh e_{j,j+1} + \sigma_{j,j+1} \sqrt{\cosh^2 e_{j,j+1} - 1} \right),$$

and (9) at  $t \rightarrow \infty$  transforms to

$$\prod_{j=1}^V \left( \cosh e_{j,j+1} + \sigma_{j,j+1} \sqrt{\cosh^2 e_{j,j+1} - 1} \right) = 1. \quad (31)$$

By the following trigonometric identity of hyperbolic geometry,  $\cosh^2 x - \sinh^2 x = 1$ , and because  $e_{j,j+1} > 0$ , we have

$$\sqrt{\cosh^2 e_{j,j+1} - 1} = \sqrt{\sinh^2 e_{j,j+1}} = \sinh e_{j,j+1}. \quad (32)$$

By (32) the equation (31) transforms to

$$\prod_{j=1}^V \left( \cosh e_{j,j+1} + \sigma_{j,j+1} \sinh e_{j,j+1} \right) = 1. \quad (33)$$

By  $\cosh x = \frac{e^x + e^{-x}}{2}$  and  $\sinh x = \frac{e^x - e^{-x}}{2}$ , we have

$$\cosh e_{j,j+1} + \sigma_{j,j+1} \sinh e_{j,j+1} = \begin{cases} e^{e_{j,j+1}} & \text{if } \sigma_{j,j+1} = 1, \\ e^{-e_{j,j+1}} & \text{if } \sigma_{j,j+1} = -1. \end{cases} = e^{\sigma_{j,j+1} e_{j,j+1}}. \quad (34)$$

Substituting (34) in (33) and taking the logarithm of the resulting equation, we get (1)  $\square$ .

The study of the behavior of the equation of flexibility (9) in neighborhoods of other singular points of the left-hand side of (9) did not give us interesting results: either we were obtaining trivial identities like  $1 = 1$  (for instance, as  $t \rightarrow \pm 1$ ), or the limit of the left-hand side of the equation of flexibility was too complicated to distinguish interesting patterns there.

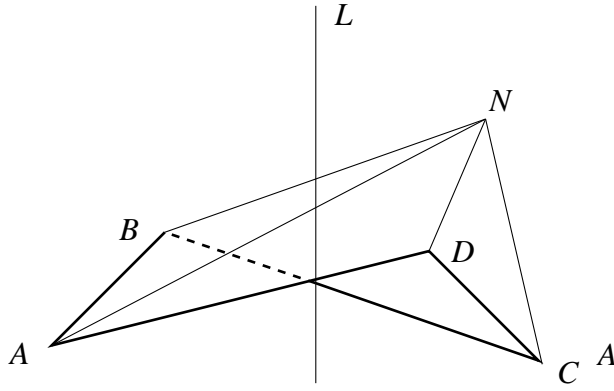


Figure 8: The construction of the Bricard-Stachel octahedron of type 1. Step 1.

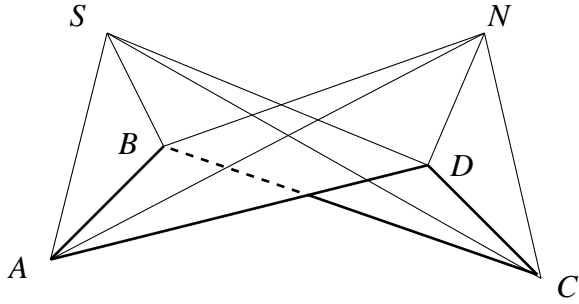


Figure 9: The construction of the Bricard-Stachel octahedron of type 1. Step 2.

## 6 Verification of the necessary flexibility condition of a non-degenerate suspension for the Bricard-Stachel octahedra in Lobachevsky 3-space

In 2002 H. Stachel [7] proved the flexibility of the analogues of the BricardTs octahedra in Lobachevsky 3-space. Let us verify the validity of the necessary flexibility condition of a nondegenerate suspension for the Bricard-Stachel octahedra in Lobachevsky 3-space.

We define an *octahedron*  $\mathcal{O}$  as the suspension  $NABCD S$  with the poles  $N$  and  $S$ , and with the vertices of the equator  $A, B, C$ , and  $D$ . Note that we can consider vertices  $A$  and  $C$  as the poles of  $\mathcal{O}$  (in this case the quadrilateral  $NDSB$  serves as the equator of  $\mathcal{O}$ ). Also we can consider vertices  $B$  and  $D$  as the poles of  $\mathcal{O}$  (in this case the quadrilateral  $NCSA$  serves as the equator of  $\mathcal{O}$ ).

### 6.1 Bricard-Stachel octahedra of types 1 and 2

The procedure of construction of the Bricard-Stachel octahedra of types 1 and 2 in Lobachevsky 3-space is the same as for the Bricard's octahedra of types 1 and 2 in Euclidean 3-space [7], [8].

Any *Bricard-Stachel octahedron of type 1* in  $\mathbb{H}^3$  can be constructed in the following way. Consider a disk-homeomorphic piece-wise linear surface  $\mathcal{S}$  in  $\mathbb{H}^3$  composed of four triangles  $ABN$ ,  $BCN$ ,  $CDN$ , and  $DAN$  such that  $d_{\mathbb{H}^3}(A, B) = d_{\mathbb{H}^3}(C, D)$  and  $d_{\mathbb{H}^3}(B, C) = d_{\mathbb{H}^3}(D, A)$ . It is known that a spatial quadrilateral  $ABCD$  which opposite sides have the same lengths, is symmetric with respect to a line  $\mathcal{L}$  passing through the middle points of its diagonals  $AC$  and  $BD$  (see Fig. 8; for a more precise analogy with the Euclidean case, in this Figure as well as in the following Figures we draw polyhedra in the Kleinian model of Lobachevsky space where lines and planes are intersections of Euclidean lines and planes with a fixed unit ball). Glue together  $\mathcal{S}$  and its symmetric image with respect to  $L$  along  $ABCD$ . Denote by  $S$  the symmetric image of  $N$  under the symmetry with respect to  $L$

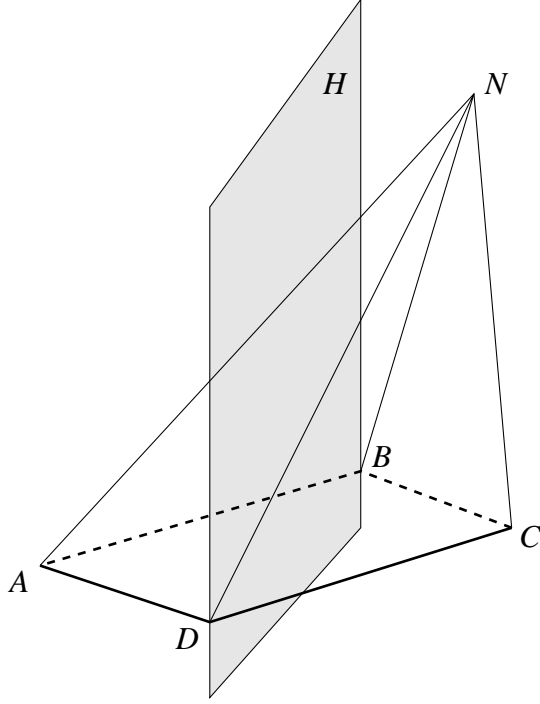


Figure 10: The construction of the Bricard-Stachel octahedron of type 2. Step 1.

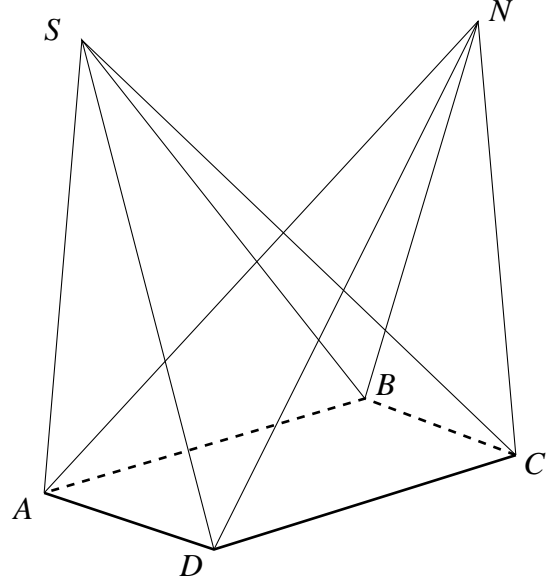


Figure 11: The construction of the Bricard-Stachel octahedron of type 2. Step 2.

(see Fig. 9). The resulting polyhedral surface  $NABCD S$  with self-intersections is flexible (because  $\mathcal{S}$  is flexible) and combinatorially it is an octahedron (according to the definition given above). We will call it a Bricard-Stachel octahedron of type 1. By construction it follows that  $d_{\mathbb{H}^3}(A, N) = d_{\mathbb{H}^3}(C, S)$ ,  $d_{\mathbb{H}^3}(B, N) = d_{\mathbb{H}^3}(D, S)$ ,  $d_{\mathbb{H}^3}(C, N) = d_{\mathbb{H}^3}(A, S)$ , and  $d_{\mathbb{H}^3}(D, N) = d_{\mathbb{H}^3}(B, S)$ .

Any *Bricard-Stachel octahedron of type 2* in  $\mathbb{H}^3$  can be constructed as follows. Consider a disk-homeomorphic piece-wise linear surface  $\mathcal{S}$  in  $\mathbb{H}^3$  composed of four triangles  $ABN$ ,  $BCN$ ,  $CDN$ , and  $DAN$  such that  $d_{\mathbb{H}^3}(A, B) = d_{\mathbb{H}^3}(B, C)$  and  $d_{\mathbb{H}^3}(C, D) = d_{\mathbb{H}^3}(D, A)$ . It is known that a spatial quadrilateral  $ABCD$  which neighbor sides at the vertices  $B$  and  $D$  have the same lengths, is symmetric with respect to a plane  $H$  which dissects the dihedral angle between the half-planes  $ABD$  and  $CBD$  (see Fig. 10). Glue together  $\mathcal{S}$  and its symmetric image with respect to  $H$  along  $ABCD$ . Denote by  $S$  the symmetric image of  $N$  under the symmetry with respect to  $H$  (see Fig. 9). The resulting polyhedral surface  $NABCD S$  with self-intersections is flexible (because  $\mathcal{S}$  is flexible) and combinatorially it is an octahedron. We will call it a Bricard-Stachel octahedron of type 2. By construction it follows that  $d_{\mathbb{H}^3}(A, N) = d_{\mathbb{H}^3}(C, S)$ ,  $d_{\mathbb{H}^3}(C, N) = d_{\mathbb{H}^3}(A, S)$ ,  $d_{\mathbb{H}^3}(B, N) = d_{\mathbb{H}^3}(B, S)$ , and  $d_{\mathbb{H}^3}(D, N) = d_{\mathbb{H}^3}(D, S)$ .

It remains to note that for every considered octahedron each of three its equators has two pairs of edges of the same lengths. Hence, the theorem 1 is valid for the Bricard-Stachel octahedra of types 1 and 2.

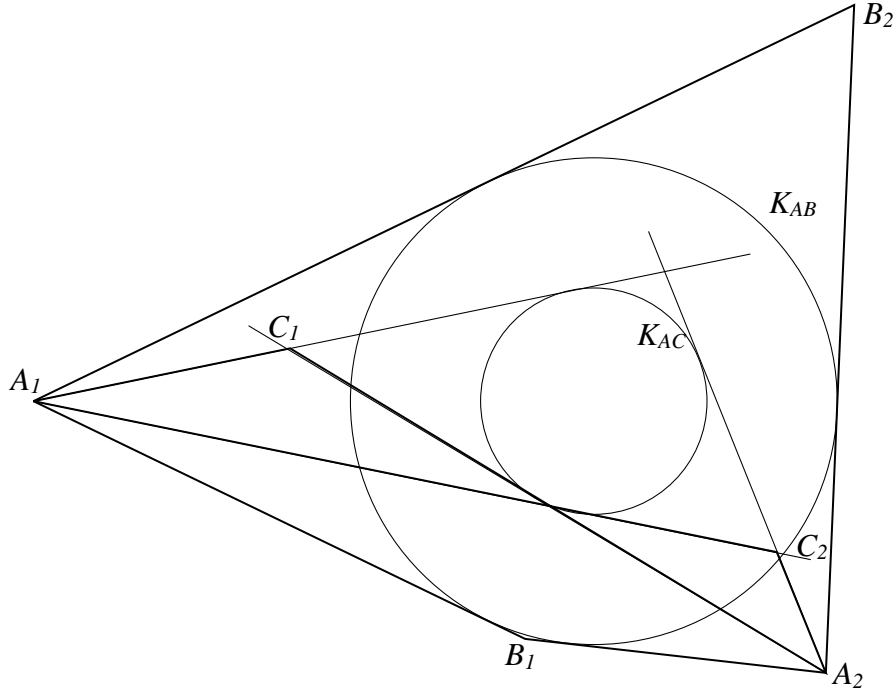


Figure 12: The construction of the Bricard-Stachel octahedron of type 3 based on circles. Step 1.

## 6.2 Bricard-Stachel octahedra of type 3

There are three subtypes of the Bricard-Stachel octahedra of type 3 in Lobachevsky space [7] which construction is based on circles, horocycles or hypercircles correspondingly. The procedure of construction is common for all subtypes of the Bricard-Stachel octahedra of type 3 and it is the same as for the Bricard's octahedra of type 3 in Euclidean space.

Any *Bricard-Stachel octahedron of type 3* in  $\mathbb{H}^3$  can be constructed in the following way. Let  $K_{AC}$  and  $K_{AB}$  be two different circles (horocycles, hypercircles) in  $\mathbb{H}^2$  with the common center  $M$  and let  $A_1, A_2$  be two different finite points outside  $K_{AC}$  and  $K_{AB}$ . In addition, suppose that  $K_{AC}, K_{AB}, A_1$  and  $A_2$  are taken in such a way that the straight lines tangent to  $K_{AB}$  and passing through  $A_1$  and  $A_2$  intersect pairwise in finite points of  $\mathbb{H}^2$  and form a quadrilateral  $A_1B_1A_2B_2$  tangent to  $K_{AB}$ ; moreover, that the straight lines tangent to  $K_{AC}$  and passing through  $A_1$  and  $A_2$  intersect pairwise in finite points of  $\mathbb{H}^2$  and form a quadrilateral  $A_1C_1A_2C_2$  tangent to  $K_{AC}$  (see Fig. 12; for clarity, we placed circles  $K_{AB}$  and  $K_{AC}$  so that their common center coincides with the center of the Kleinian model of Lobachevsky space. In this case  $K_{AB}$  and  $K_{AC}$  are Euclidean circles as well). A polyhedron  $\mathcal{O}$  with the vertices  $A_i, B_j, C_k$ , with the edges  $A_iB_j, A_iC_k, B_jC_k$ , and with the faces  $\triangle A_iB_jC_k, i, j, k \in \{1, 2\}$ , is an octahedron in the sense of the definition given above (see Fig. 13). The following pairs of vertices can serve as the poles of  $\mathcal{O}$ :  $(A_1, A_2)$  with the corresponding equator  $B_1C_1B_2C_2$ ,  $(B_1, B_2)$  with the equator  $A_1C_1A_2C_2$ , and  $(C_1, C_2)$  with the equator  $A_1B_1A_2B_2$ . Suppose in addition that  $\mathcal{O}$  does not have symmetries. We will call such octahedron  $\mathcal{O}$  a Bricard-Stachel octahedron of type 3.

According to H. Stachel [7],  $\mathcal{O}$  flexes continuously in  $\mathbb{H}^3$ . Moreover,  $\mathcal{O}$  admits two flat

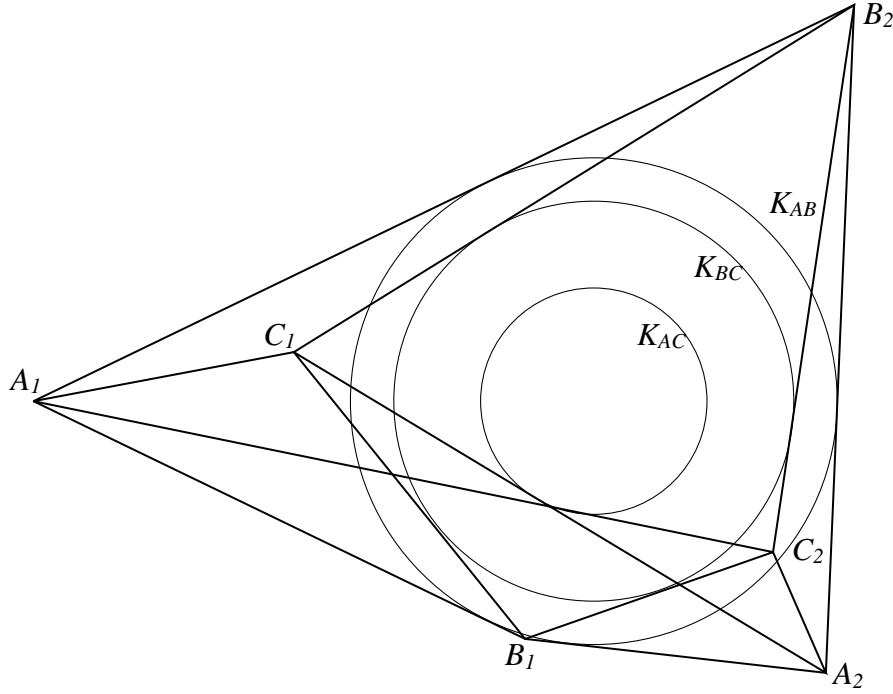


Figure 13: The construction of the Bricard-Stachel octahedron of type 3 based on circles. Step 2.

positions during the flex (we constructed  $\mathcal{O}$  in one of its flat positions). Hence, for every equator of  $\mathcal{O}$ ,  $A_1B_1A_2B_2$ ,  $B_1C_1B_2C_2$ , and  $A_1C_1A_2C_2$ , all straight lines containing a side of the equator are tangent to some circle (horocycle, hypercircle) at least in one flat position of  $\mathcal{O}$ . Using this fact, we will prove that the theorem 1 is valid for the Bricard-Stachel octahedra of type 3. We have to consider three possible cases: when an equator of  $\mathcal{O}$  is tangent to a circle, to a horocycle, or to a hypercircle in  $\mathbb{H}^2$ . Here we study the most common situation when any three vertices of an equator of a flexible octahedron in its flat position do not lie on a straight line.

### 6.2.1 An equator of a Bricard-Stachel octahedron of type 3 is tangent to a circle in $\mathbb{H}^2$

Let  $M$  be the center of the circle  $K_{AB}$  with the radius  $R$  in  $\mathbb{H}^2$  and let all straight lines containing a side of the quadrilateral  $A_1B_1A_2B_2$  are tangent to  $K_{AB}$ . Let us draw the segments  $MP_1$ ,  $MP_2$ ,  $MP_3$ ,  $MP_4$  connecting  $M$  with the straight lines  $A_1B_2$ ,  $A_2B_2$ ,  $A_2B_1$ ,  $A_1B_1$  and perpendicular to the corresponding lines. By construction,  $d_{\mathbb{H}^2}(M, P_1) = d_{\mathbb{H}^2}(M, P_2) = d_{\mathbb{H}^2}(M, P_3) = d_{\mathbb{H}^2}(M, P_4) = R$ .

By the Pythagorean theorem for Lobachevsky space [9] applied to  $\triangle A_1MP_1$  and  $\triangle A_1MP_4$ , we obtain:  $\cosh d_{\mathbb{H}^2}(A_1, P_1) = \cosh d_{\mathbb{H}^2}(A_1, P_4) = \cosh d_{\mathbb{H}^2}(A_1, M) / \cosh R$ . Then  $a \stackrel{\text{def}}{=} d_{\mathbb{H}^2}(A_1, P_1) = d_{\mathbb{H}^2}(A_1, P_4)$ . Similarly we get:  $b \stackrel{\text{def}}{=} d_{\mathbb{H}^2}(B_2, P_1) = d_{\mathbb{H}^2}(B_2, P_2)$ ,  $c \stackrel{\text{def}}{=} d_{\mathbb{H}^2}(A_2, P_2) = d_{\mathbb{H}^2}(A_2, P_3)$ , and  $d \stackrel{\text{def}}{=} d_{\mathbb{H}^2}(B_1, P_3) = d_{\mathbb{H}^2}(B_1, P_4)$ .

If the circle  $K_{AB}$  is inscribed in the quadrilateral  $A_1B_1A_2B_2$  (see Fig. 12), then  $d_{\mathbb{H}^2}(A_1, B_2) = a + b$ ,  $d_{\mathbb{H}^2}(A_2, B_2) = b + c$ ,  $d_{\mathbb{H}^2}(A_2, B_1) = c + d$ ,  $d_{\mathbb{H}^2}(A_1, B_1) = a + d$ ,

and the identity

$$d_{\mathbb{H}^2}(A_1, B_2) - d_{\mathbb{H}^2}(A_2, B_2) + d_{\mathbb{H}^2}(A_1, B_1) - d_{\mathbb{H}^2}(A_2, B_1) = 0 \quad (35)$$

holds true.

If the circle  $K_{AB}$  is tangent to the quadrilateral  $A_1B_1A_2B_2$  externally (this case corresponds to the quadrilateral  $A_1C_1A_2C_2$  and to the circle  $K_{AC}$  in the Fig. 12), then  $d_{\mathbb{H}^2}(A_1, B_2) = a - b$ ,  $d_{\mathbb{H}^2}(A_2, B_2) = b + c$ ,  $d_{\mathbb{H}^2}(A_2, B_1) = c - d$ ,  $d_{\mathbb{H}^2}(A_1, B_1) = a + d$ , and the identity

$$d_{\mathbb{H}^2}(A_1, B_2) + d_{\mathbb{H}^2}(A_2, B_2) - d_{\mathbb{H}^2}(A_1, B_1) - d_{\mathbb{H}^2}(A_2, B_1) = 0 \quad (36)$$

holds true.

By (35) and (36), the theorem 1 is valid for any equator of a Bricard-Stachel octahedron of type 3 tangent to a circle in at least one of its flat positions.

### 6.2.2 An equator of a Bricard-Stachel octahedron of type 3 is tangent to a horocycle in $\mathbb{H}^2$

Let us consider the Poincaré upper half-plane model of Lobachevsky plane  $\mathbb{H}^2$  with the coordinates  $(\rho, z)$  (i.e., with the metric given by the formula  $ds^2 = \frac{d\rho^2 + dz^2}{z^2}$ ). Without loss of generality we can assume that the center of the horocycle tangent to the equator of a Bricard-Stachel octahedron  $\mathcal{O}$  of type 3, coincides with the (unique) point  $\infty$  at infinity of  $\mathbb{H}^2$  which does not lie on the Euclidean line  $z = 0$ . We denote the family of such horocycles by  $K = \{\rho = R | R > 0\}$ . Let  $K_R \in K$  and let  $A_1 = (\rho_{A_1}, z_{A_1})$  and  $A_2 = (\rho_{A_2}, z_{A_2})$  be two opposite vertices of  $\mathcal{O}$ , such that the straight line (in  $\mathbb{H}^2$ ) passing through  $A_1$  and  $A_2$  is not tangent to  $K_R$ . All the vertices of  $\mathcal{O}$  are located outside  $K_R$ , hence  $z_{A_1} < R$  and  $z_{A_2} < R$ . We will construct all possible quadrangles tangent to  $K_R$  with the opposite vertices  $A_1$  and  $A_2$ , i.e., all quadrangles that can serve as equators of  $\mathcal{O}$ . Then we will verify the validity of the theorem 1 for such quadrangles.

Let  $T = (\rho_T, z_T)$  be a point in  $\mathbb{H}^2$  and let  $\Lambda$  be a straight line in  $\mathbb{H}^2$  passing through  $T$  which is realized in the Poincaré upper half-plane as the Euclidean demi-circle with the radius  $\sqrt{(\rho_T - \rho_{T,\Lambda})^2 + z_T^2}$  and with the center  $O_\Lambda^T = (\rho_{T,\Lambda}, 0)$ . Then the angle  $\varphi_T^\Lambda \stackrel{\text{def}}{=} \angle TO_\Lambda^T \rho \in (0, \pi)$  determines uniquely a position of  $T$  on  $\Lambda$ .

**Remark 1** For every finite point  $T = (\rho_T, z_T)$ ,  $z_T < R$ , there exist precisely two straight lines  $\Lambda_l^T$  and  $\Lambda_r^T$  tangent to the horocycle  $K_R$  and containing  $T$ . They are realized in the Poincaré upper half-plane as the Euclidean demi-circles with the radius  $R$  and with the centers  $O_l^T = (\rho_{T,l}, 0)$  and  $O_r^T = (\rho_{T,r}, 0)$ ,  $\rho_{T,l} \leq \rho_T \leq \rho_{T,r}$ . The angles  $\varphi_T^l \stackrel{\text{def}}{=} \angle TO_l^T \rho$  and  $\varphi_T^r \stackrel{\text{def}}{=} \angle TO_r^T \rho$  serve as the coordinates of  $T$  on  $\Lambda_l^T$  and  $\Lambda_r^T$  correspondingly. Then, by construction, we get:  $\varphi_T^r = \pi - \varphi_T^l$ . Hence,

$$\cos \varphi_T^r = -\cos \varphi_T^l. \quad (37)$$

According to the remark 1, there are two straight lines,  $\Lambda_l^{A_1}$ , and  $\Lambda_r^{A_1}$ , passing through  $A_1$  and tangent to  $K_R$ , which are realised in  $\mathbb{H}^2$  as the Euclidean demi-circles with the

radius  $R$  and with the centers  $O_l^{A_1} = (\rho_{A_1,l}, 0)$ ,  $O_r^{A_1} = (\rho_{A_1,r}, 0)$ ,  $\rho_{A_1,l} \leq \rho_{A_1} \leq \rho_{A_1,r}$ . The angles  $\varphi_{A_1}^{\Lambda_l^{A_1}} \stackrel{\text{def}}{=} \angle A_1 O_l^{A_1} \rho$ ,  $\varphi_{A_1}^{\Lambda_r^{A_1}} \stackrel{\text{def}}{=} \angle A_1 O_r^{A_1} \rho$  serve as the coordinates of  $A_1$  on  $\Lambda_l^{A_1}$  and  $\Lambda_r^{A_1}$  correspondingly. Moreover,

$$\cos \varphi_{A_1}^{\Lambda_r^{A_1}} = -\cos \varphi_{A_1}^{\Lambda_l^{A_1}}. \quad (38)$$

Similarly, there are two straight lines,  $\Lambda_l^{A_2}$ , and  $\Lambda_r^{A_2}$ , passing through  $A_2$  and tangent to  $K_R$ , which are realised in  $\mathbb{H}^2$  as the Euclidean demi-circles with the radius  $R$  and with the centers  $O_l^{A_2} = (\rho_{A_2,l}, 0)$ ,  $O_r^{A_2} = (\rho_{A_2,r}, 0)$ ,  $\rho_{A_2,l} \leq \rho_{A_2} \leq \rho_{A_2,r}$ . The angles  $\varphi_{A_2}^{\Lambda_l^{A_2}} \stackrel{\text{def}}{=} \angle A_2 O_l^{A_2} \rho$ ,  $\varphi_{A_2}^{\Lambda_r^{A_2}} \stackrel{\text{def}}{=} \angle A_2 O_r^{A_2} \rho$  serve as the coordinates of  $A_2$  on  $\Lambda_l^{A_2}$  and  $\Lambda_r^{A_2}$  correspondingly. Moreover,

$$\cos \varphi_{A_2}^{\Lambda_r^{A_2}} = -\cos \varphi_{A_2}^{\Lambda_l^{A_2}}. \quad (39)$$

Suppose that  $\Lambda_l^{A_1}$  and  $\Lambda_l^{A_2}$  intersect at a point  $B_1$ . Then the angles  $\varphi_{B_1}^{\Lambda_l^{A_1}} \stackrel{\text{def}}{=} \angle B_1 O_l^{A_1} \rho$ ,  $\varphi_{B_1}^{\Lambda_l^{A_2}} \stackrel{\text{def}}{=} \angle B_1 O_l^{A_2} \rho$  serve as the coordinates of  $B_1$  on  $\Lambda_l^{A_1}$  and  $\Lambda_l^{A_2}$  correspondingly. Moreover,

$$\cos \varphi_{B_1}^{\Lambda_l^{A_2}} = -\cos \varphi_{B_1}^{\Lambda_l^{A_1}}. \quad (40)$$

Also suppose that  $\Lambda_r^{A_1}$  and  $\Lambda_r^{A_2}$  intersect at a point  $B_2$ . Then the angles  $\varphi_{B_2}^{\Lambda_r^{A_1}} \stackrel{\text{def}}{=} \angle B_2 O_r^{A_1} \rho$ ,  $\varphi_{B_2}^{\Lambda_r^{A_2}} \stackrel{\text{def}}{=} \angle B_2 O_r^{A_2} \rho$  serve as the coordinates of  $B_2$  on  $\Lambda_r^{A_1}$  and  $\Lambda_r^{A_2}$  correspondingly. Moreover,

$$\cos \varphi_{B_2}^{\Lambda_r^{A_2}} = -\cos \varphi_{B_2}^{\Lambda_r^{A_1}}. \quad (41)$$

Let the straight lines  $\Lambda_r^{A_1}$  and  $\Lambda_l^{A_2}$  intersect at a point  $C_1$ . Then the angles  $\varphi_{C_1}^{\Lambda_r^{A_1}} \stackrel{\text{def}}{=} \angle C_1 O_r^{A_1} \rho$ ,  $\varphi_{C_1}^{\Lambda_l^{A_2}} \stackrel{\text{def}}{=} \angle C_1 O_l^{A_2} \rho$  serve as the coordinates of  $C_1$  on  $\Lambda_r^{A_1}$  and  $\Lambda_l^{A_2}$  correspondingly. Moreover,

$$\cos \varphi_{C_1}^{\Lambda_l^{A_2}} = -\cos \varphi_{C_1}^{\Lambda_r^{A_1}}. \quad (42)$$

Also, let the straight lines  $\Lambda_l^{A_1}$  and  $\Lambda_r^{A_2}$  intersect at a point  $C_2$ . Then the angles  $\varphi_{C_2}^{\Lambda_l^{A_1}} \stackrel{\text{def}}{=} \angle C_2 O_l^{A_1} \rho$ ,  $\varphi_{C_2}^{\Lambda_r^{A_2}} \stackrel{\text{def}}{=} \angle C_2 O_r^{A_2} \rho$  serve as the coordinates of  $C_2$  on  $\Lambda_l^{A_1}$  and  $\Lambda_r^{A_2}$  correspondingly. Moreover,

$$\cos \varphi_{C_2}^{\Lambda_r^{A_2}} = -\cos \varphi_{C_2}^{\Lambda_l^{A_1}}. \quad (43)$$

By construction, the quadrangles  $A_1 B_1 A_2 B_2$  and  $A_1 C_1 A_2 C_2$  are tangent to  $K_R$ , and the points  $A_1$ ,  $A_2$  are opposite vertices of each of these quadrangles. In order to verify the validity of the theorem 1 for the flexible octahedra with the equator  $A_1 B_1 A_2 B_2$  or  $A_1 C_1 A_2 C_2$  we need to prove the following easy statement.

**Lemma 3** *Given a Poincaré upper half-plane  $\mathbb{H}^2$  with the coordinates  $(\rho, z)$  (i.e., with the metric given by the formula  $ds^2 = \frac{d\rho^2 + dz^2}{z^2}$ ). Let  $A$  and  $B$  be points on the straight line  $\Lambda$  realised in  $\mathbb{H}^2$  as the Euclidean demi-circle with the radius  $R$  and with the center  $O_\Lambda = (\rho_{O_\Lambda}, 0)$ , and let the angles  $\varphi_A \stackrel{\text{def}}{=} \angle A O_\Lambda \rho$ ,  $\varphi_B \stackrel{\text{def}}{=} \angle B O_\Lambda \rho$  serve as the coordinates of  $A$  and*



$B$  correspondingly on  $\Lambda$ . Also we assume that  $0 < \varphi_A \leq \phi_B < \pi$ . Then the distance between  $A$  and  $B$  is calculated as follows:

$$d_{\mathbb{H}^2}(A, B) = \frac{1}{2} \ln \left[ \left( \frac{1 + \cos \varphi_A}{1 + \cos \varphi_B} \right) \left( \frac{1 - \cos \varphi_B}{1 - \cos \varphi_A} \right) \right]. \quad (44)$$

**Proof.** The hyperbolic segment  $\Lambda_{AB}$  connecting the points  $A$  and  $B$  is specified parametrically by the formulas  $\Lambda_{AB}(t) : (\rho(\varphi), z(\varphi))$ ,  $\varphi \in [\varphi_A, \varphi_B]$ , where  $\rho(\varphi) = \rho_{O_\Lambda} + R \cos \varphi$ ,  $z(\varphi) = R \sin \varphi$ ,  $A = \Lambda_{AB}(\varphi_A)$ ,  $B = \Lambda_{AB}(\varphi_B)$ . The direct calculation shows that the lengths of  $\Lambda_{AB}$  is equal to the right-hand side of (44).  $\square$

By Lemma 3, the lengths of the edges of the quadrilateral  $A_1 B_1 A_2 B_2$  are calculated as follows:

$$d_{\mathbb{H}^2}(A_1, B_1) = \frac{1}{2} \ln \left[ \left( \frac{1 + \cos \varphi_{A_1}^{\Lambda_l^{A_1}}}{1 + \cos \varphi_{B_1}^{\Lambda_l^{A_1}}} \right) \left( \frac{1 - \cos \varphi_{B_1}^{\Lambda_l^{A_1}}}{1 - \cos \varphi_{A_1}^{\Lambda_l^{A_1}}} \right) \right], \quad (45)$$

$$d_{\mathbb{H}^2}(A_2, B_1) = \frac{1}{2} \ln \left[ \left( \frac{1 + \cos \varphi_{A_2}^{\Lambda_l^{A_2}}}{1 + \cos \varphi_{B_1}^{\Lambda_l^{A_2}}} \right) \left( \frac{1 - \cos \varphi_{B_1}^{\Lambda_l^{A_2}}}{1 - \cos \varphi_{A_2}^{\Lambda_l^{A_2}}} \right) \right], \quad (46)$$

$$d_{\mathbb{H}^2}(B_2, A_1) = \frac{1}{2} \ln \left[ \left( \frac{1 + \cos \varphi_{B_2}^{\Lambda_r^{A_1}}}{1 + \cos \varphi_{A_1}^{\Lambda_r^{A_1}}} \right) \left( \frac{1 - \cos \varphi_{A_1}^{\Lambda_r^{A_1}}}{1 - \cos \varphi_{B_2}^{\Lambda_r^{A_1}}} \right) \right], \quad (47)$$

$$d_{\mathbb{H}^2}(B_2, A_2) = \frac{1}{2} \ln \left[ \left( \frac{1 + \cos \varphi_{B_2}^{\Lambda_r^{A_2}}}{1 + \cos \varphi_{A_2}^{\Lambda_r^{A_2}}} \right) \left( \frac{1 - \cos \varphi_{A_2}^{\Lambda_r^{A_2}}}{1 - \cos \varphi_{B_2}^{\Lambda_r^{A_2}}} \right) \right]. \quad (48)$$

Then, by (38)—(41), we get:

$$d_{\mathbb{H}^2}(A_1, B_1) + d_{\mathbb{H}^2}(A_2, B_1) - d_{\mathbb{H}^2}(B_2, A_1) - d_{\mathbb{H}^2}(B_2, A_2) = 0. \quad (49)$$

By Lemma 3, the lengths of the edges of the quadrilateral  $A_1 C_1 A_2 C_2$  are calculated as follows:

$$d_{\mathbb{H}^2}(C_1, A_1) = \frac{1}{2} \ln \left[ \left( \frac{1 + \cos \varphi_{C_1}^{\Lambda_r^{A_1}}}{1 + \cos \varphi_{A_1}^{\Lambda_r^{A_1}}} \right) \left( \frac{1 - \cos \varphi_{A_1}^{\Lambda_r^{A_1}}}{1 - \cos \varphi_{C_1}^{\Lambda_r^{A_1}}} \right) \right], \quad (50)$$

$$d_{\mathbb{H}^2}(C_2, A_1) = \frac{1}{2} \ln \left[ \left( \frac{1 + \cos \varphi_{C_2}^{\Lambda_l^{A_1}}}{1 + \cos \varphi_{A_1}^{\Lambda_l^{A_1}}} \right) \left( \frac{1 - \cos \varphi_{A_1}^{\Lambda_l^{A_1}}}{1 - \cos \varphi_{C_2}^{\Lambda_l^{A_1}}} \right) \right], \quad (51)$$

$$d_{\mathbb{H}^2}(A_2, C_1) = \frac{1}{2} \ln \left[ \left( \frac{1 + \cos \varphi_{A_2}^{\Lambda_l^{A_2}}}{1 + \cos \varphi_{C_1}^{\Lambda_l^{A_2}}} \right) \left( \frac{1 - \cos \varphi_{C_1}^{\Lambda_l^{A_2}}}{1 - \cos \varphi_{A_2}^{\Lambda_l^{A_2}}} \right) \right], \quad (52)$$

$$d_{\mathbb{H}^2}(A_2, C_2) = \frac{1}{2} \ln \left[ \left( \frac{1 + \cos \varphi_{A_2}^{\Lambda_r^{A_2}}}{1 + \cos \varphi_{C_2}^{\Lambda_r^{A_2}}} \right) \left( \frac{1 - \cos \varphi_{C_2}^{\Lambda_r^{A_2}}}{1 - \cos \varphi_{A_2}^{\Lambda_r^{A_2}}} \right) \right]. \quad (53)$$

By (38), (39), (42), and (43), it is easy to verify that

$$d_{\mathbb{H}^2}(C_2, A_1) + d_{\mathbb{H}^2}(C_1, A_1) - d_{\mathbb{H}^2}(A_2, C_1) - d_{\mathbb{H}^2}(A_2, C_2) = 0. \quad (54)$$

According to (49) and (54), the theorem 1 is valid for any equator of a Bricard-Stachel octahedron of type 3 tangent to a horocycle in at least one of its flat positions.

### 6.2.3 An equator of a Bricard-Stachel octahedron of type 3 is tangent to a hypercircle in $\mathbb{H}^2$

Let us consider the Poincaré upper half-plane model of Lobachevsky plane  $\mathbb{H}^2$  with the coordinates  $(\rho, z)$  (i.e., with the metric given by the formula  $ds^2 = \frac{d\rho^2 + dz^2}{z^2}$ ). Without loss of generality we can assume that the hypercircle tangent to the equator of a Bricard-Stachel octahedron  $\mathcal{O}$  of type 3, passes through the (unique) point  $\infty$  at infinity of  $\mathbb{H}^2$  which does not lie on the Euclidean line  $z = 0$ , and through the point  $O = (0, 0)$  at infinity of  $\mathbb{H}^2$ . Every such hypercircle is specified by the equation  $z = \rho \tan \alpha$  for some  $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ . By the symmetry of  $\mathbb{H}^2$  with respect to the straight line  $\rho = 0$ , it is sufficient to consider the family of hypercircles  $K = \{z = \rho \tan \alpha | \alpha \in (0, \frac{\pi}{2})\}$ . Let  $K_\alpha \in K$ . We will construct all possible quadrangles tangent to  $K_\alpha$  such that none of their vertices belongs to  $K_\alpha$ , i.e., all quadrangles that can serve as equators of  $\mathcal{O}$ . Then we will verify the validity of the theorem 1 for such quadrangles.

Let us study the quadrangles based on the straight lines  $\Lambda_l^{A_1}, \Lambda_r^{A_1}, \Lambda_l^{A_2}, \Lambda_r^{A_2}$ , tangent to  $K_\alpha$ , which are realised in  $\mathbb{H}^2$  as the Euclidean demi-circles with the centers  $O_l^{A_1} = (\rho_{A_1,l}, 0)$ ,  $O_r^{A_1} = (\rho_{A_1,r}, 0)$ ,  $O_l^{A_2} = (\rho_{A_2,l}, 0)$ ,  $O_r^{A_2} = (\rho_{A_2,r}, 0)$ . Also, let  $\Lambda_l^{A_1}$  and  $\Lambda_r^{A_1}$  intersect at a point  $A_1$ ,  $\Lambda_l^{A_2}$  and  $\Lambda_r^{A_2}$  intersect at a point  $A_2$ . Assume that  $A_1$  and  $A_2$  are two opposite vertices of  $\mathcal{O}$ , and that the inequalities  $0 < \rho_{A_1,l} < \rho_{A_1,r}$ ,  $0 < \rho_{A_2,l} < \rho_{A_2,r}$  hold true.

**Remark 2** Let  $T = (\rho_T, z_T)$  be a point in  $\mathbb{H}^2$ , which serves as the intersection of straight lines  $\Lambda_l^T$  and  $\Lambda_r^T$  tangent to a hypercircle  $K_\alpha$ , and let  $\Lambda_l^T$  and  $\Lambda_r^T$  are realised in  $\mathbb{H}^2$  as the Euclidean demi-circles with the centers  $O_l^T = (\rho_{T,l}, 0)$ ,  $O_r^T = (\rho_{T,r}, 0)$  ( $\rho_{T,l} < \rho_{T,r}$ ). Then, by Remark 1, the angles  $\varphi_T^l \stackrel{\text{def}}{=} \angle TO_l^T \rho$  and  $\varphi_T^r \stackrel{\text{def}}{=} \angle TO_r^T \rho$  determine uniquely the positions of  $T$  on  $\Lambda_l^T$  and  $\Lambda_r^T$  correspondingly. Moreover,

$$\cos \varphi_T^l = \frac{\rho_{T,r}}{\rho_{T,l}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2} \quad \text{and} \quad \cos \varphi_T^r = \frac{\rho_{T,l}}{\rho_{T,r}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2}. \quad (55)$$

**Proof.**  $\Lambda_l^T$  and  $\Lambda_r^T$  are tangent to  $K_\alpha$ . Hence, the radii  $R_l$  and  $R_r$  of the demi-circles realizing  $\Lambda_l^T$  and  $\Lambda_r^T$  in  $\mathbb{H}^2$  are determined by the formulas

$$R_l = \rho_{T,l} \sin \alpha \quad \text{and} \quad R_r = \rho_{T,r} \sin \alpha. \quad (56)$$

Let  $T_\infty$  be a point with coordinates  $(\rho_T, 0)$ . Applying the Euclidean Pythagorean theorem to  $\triangle TT_\infty O_r^T$  and simplifying the obtained expression, we get:

$$\rho_T^2 + z_T^2 = 2\rho_T \rho_{T,l} - \rho_{T,l}^2 \cos^2 \alpha. \quad (57)$$

Similarly, from  $\triangle TT_\infty O_l^T$  we get that

$$\rho_T^2 + z_T^2 = 2\rho_T \rho_{T,r} - \rho_{T,r}^2 \cos^2 \alpha. \quad (58)$$

Subtracting (57) from (58), we easily deduce:

$$\rho_T = \frac{\rho_{T,r} + \rho_{T,l}}{2} \cos^2 \alpha. \quad (59)$$

From the definitions of the cosines of  $\varphi_T^l$  and  $\varphi_T^r$  ( $\cos \varphi_T^l = (\rho_T - \rho_{T,l})/R_l$  and  $\cos \varphi_T^r = (\rho_T - \rho_{T,r})/R_r$ ), taking into account (56) and (59), we obtain (55).  $\square$

By Remark 2, the angles  $\varphi_{A_1}^{\Lambda_l^{A_1}} \stackrel{\text{def}}{=} \angle A_1 O_l^{A_1} \rho$  and  $\varphi_{A_1}^{\Lambda_r^{A_1}} \stackrel{\text{def}}{=} \angle A_1 O_r^{A_1} \rho$  determine uniquely the positions of  $A_1$  on  $\Lambda_l^{A_1}$  and  $\Lambda_r^{A_1}$  correspondingly. Moreover,

$$\cos \varphi_{A_1}^{\Lambda_l^{A_1}} = \frac{\rho_{A_1,r}}{\rho_{A_1,l}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2} \quad \text{and} \quad \cos \varphi_{A_1}^{\Lambda_r^{A_1}} = \frac{\rho_{A_1,l}}{\rho_{A_1,r}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2}. \quad (60)$$

Similarly, the angles  $\varphi_{A_2}^{\Lambda_l^{A_2}} \stackrel{\text{def}}{=} \angle A_2 O_l^{A_2} \rho$  and  $\varphi_{A_2}^{\Lambda_r^{A_2}} \stackrel{\text{def}}{=} \angle A_2 O_r^{A_2} \rho$  serve as the coordinates of  $A_2$  on  $\Lambda_l^{A_2}$  and  $\Lambda_r^{A_2}$  correspondingly. Moreover,

$$\cos \varphi_{A_2}^{\Lambda_l^{A_2}} = \frac{\rho_{A_2,r}}{\rho_{A_2,l}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2} \quad \text{and} \quad \cos \varphi_{A_2}^{\Lambda_r^{A_2}} = \frac{\rho_{A_2,l}}{\rho_{A_2,r}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2}. \quad (61)$$

Suppose that the straight lines  $\Lambda_l^{A_1}$  and  $\Lambda_l^{A_2}$  intersect at a point  $B_1$ . Then the angles  $\varphi_{B_1}^{\Lambda_l^{A_1}} \stackrel{\text{def}}{=} \angle B_1 O_l^{A_1} \rho$  and  $\varphi_{B_1}^{\Lambda_l^{A_2}} \stackrel{\text{def}}{=} \angle B_1 O_l^{A_2} \rho$  serve as the coordinates of  $B_1$  on  $\Lambda_l^{A_1}$  and  $\Lambda_l^{A_2}$  correspondingly. Moreover,

$$\cos \varphi_{B_1}^{\Lambda_l^{A_1}} = \frac{\rho_{A_2,l}}{\rho_{A_1,l}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2} \quad \text{and} \quad \cos \varphi_{B_1}^{\Lambda_l^{A_2}} = \frac{\rho_{A_1,l}}{\rho_{A_2,l}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2}. \quad (62)$$

Suppose also that  $\Lambda_r^{A_1}$  and  $\Lambda_r^{A_2}$  intersect at a point  $B_2$ . Then the angles  $\varphi_{B_2}^{\Lambda_r^{A_1}} \stackrel{\text{def}}{=} \angle B_2 O_r^{A_1} \rho$  and  $\varphi_{B_2}^{\Lambda_r^{A_2}} \stackrel{\text{def}}{=} \angle B_2 O_r^{A_2} \rho$  serve as the coordinates of  $B_2$  on  $\Lambda_r^{A_1}$  and  $\Lambda_r^{A_2}$  correspondingly. Moreover,

$$\cos \varphi_{B_2}^{\Lambda_r^{A_1}} = \frac{\rho_{A_2,r}}{\rho_{A_1,r}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2} \quad \text{and} \quad \cos \varphi_{B_2}^{\Lambda_r^{A_2}} = \frac{\rho_{A_1,r}}{\rho_{A_2,r}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2}. \quad (63)$$

Suppose that  $\Lambda_r^{A_1}$  and  $\Lambda_l^{A_2}$  intersect at a point  $C_1$ . Then the angles  $\varphi_{C_1}^{\Lambda_r^{A_1}} \stackrel{\text{def}}{=} \angle C_1 O_r^{A_1} \rho$  and  $\varphi_{C_1}^{\Lambda_l^{A_2}} \stackrel{\text{def}}{=} \angle C_1 O_l^{A_2} \rho$  serve as the coordinates of  $C_1$  on  $\Lambda_r^{A_1}$  and  $\Lambda_l^{A_2}$  correspondingly. Moreover,

$$\cos \varphi_{C_1}^{\Lambda_l^{A_2}} = \frac{\rho_{A_1,r}}{\rho_{A_2,l}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2} \quad \text{and} \quad \cos \varphi_{C_1}^{\Lambda_r^{A_1}} = \frac{\rho_{A_2,l}}{\rho_{A_1,r}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2}. \quad (64)$$

Suppose also that  $\Lambda_l^{A_1}$  and  $\Lambda_r^{A_2}$  intersect at a point  $C_2$ . Then the angles  $\varphi_{C_2}^{\Lambda_l^{A_1}} \stackrel{\text{def}}{=} \angle C_2 O_l^{A_1} \rho$  and  $\varphi_{C_2}^{\Lambda_r^{A_2}} \stackrel{\text{def}}{=} \angle C_2 O_r^{A_2} \rho$  serve as the coordinates of  $C_2$  on  $\Lambda_l^{A_1}$  and  $\Lambda_r^{A_2}$  correspondingly. Moreover,

$$\cos \varphi_{C_2}^{\Lambda_l^{A_1}} = \frac{\rho_{A_2,r}}{\rho_{A_1,l}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2} \quad \text{and} \quad \cos \varphi_{C_2}^{\Lambda_r^{A_2}} = \frac{\rho_{A_1,l}}{\rho_{A_2,r}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2}. \quad (65)$$

As in the case of the quadrangles tangent to a horocycle in  $\mathbb{H}^2$ , the lengths of the edges of  $A_1 B_1 A_2 B_2$  are expressed in (45)–(48), and the lengths of the edges of  $A_1 C_1 A_2 C_2$  are calculated in (50)–(53). Taking into account (60)–(65), it is easy to state the validity of (49) and (54).

According to (49) and (54), the theorem 1 is valid for any equator of a Bricard-Stachel octahedron of type 3 tangent to a hypercircle in at least one of its flat positions.

The case when three vertices of an equator of a flexible octahedron in its flat position lie on a straight line, is similar. The case when all four vertices of an equator lie on a straight line, is trivial.

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Dmitriy Slutskiy  
 Sobolev Institute of Mathematics of the SB RAS,  
 4 Acad. Koptyug avenue, 630090 Novosibirsk, Russia  
 and  
 Novosibirsk State University,  
 2 Pirogova Street, 630090, Novosibirsk, Russia  
 slutski@ngs.ru